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Short Communication

Examination of the fundamental frequencies of annular plates with small core

C.Y. Wang[a,*], C.M. Wang[b]

[a]Department of Mathematics, Michigan State University, Wells Hall, East Lansing, MI 48824, USA
[b]Department of Civil Engineering, National University of Singapore, Kent Ridge, Singapore 119260, Singapore

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1. Introduction

The natural frequencies of vibrating annular plates have been reported in numerous sources [1-6]. Invariably a table is presented, with core radii b (normalized by the plate diameter) ranging from 0.1 to 0.9. Smaller core radii are not treated as numerical techniques suffer from severe scaling problems.

The fundamental frequency (below which no vibration would occur) corresponds in general to the axisymmetric mode with no nodal diameter, except for the plate with both edges free, which vibrates with two nodal diameters. It has been noted, however, that the fundamental frequency may switch from no nodal diameter to one as the core radius is decreased for annular plates with their inner edge either clamped or simply supported and the outer edge free [7-9]. The non-axisymmetric fundamental mode of vibration for annular plates with very small-sized supported cores was not recognized by Olhoff [10] as evidence from his assumption of an axisymmetric vibration mode when seeking the optimal design of a centrally supported plate for maximum fundamental frequency. The analytical nature of the clamped-free annular plate case was first studied by Southwell [11], who employed asymptotic expansions as b shrinks to zero. It was found that the frequency rises singularly from zero for small b.

*Corresponding author. Fax: +1-517-432-1562.
E-mail address: cywang@mth.msu.edu (C.Y. Wang).

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The purpose of the present study is to examine the fundamental frequencies of vibrating annular plates with small cores, especially for b less than 0.1 where hardly any frequency values are reported. As small core problems are important in situations such as the control of vibrations by bolting or nailing the interior of a plate, this study will answer the following questions: Is the fundamental frequency finite or zero as $b \to 0$? What is the frequency for small but finite b? At what core radius does the mode switches from a non-axisymmetric mode to an axisymmetric one? How does the boundary conditions and the Poisson ratio affect this transition core radius?

2. Formulation

Consider a thin, annular plate with constant thickness h, outer radius R and inner radius bR. The plate will be modeled using the classical thin plate theory. In carrying out the free vibration analysis of the annular plate, the transverse displacement can be separated as $w(r)\cos(n\theta)e^{i\omega t}$ where r is the radial distance normalized by R, n is the number of nodal diameters, ω is the angular frequency, and w is the transverse displacement and it is given by a linear combination of the Bessel functions $J_n(kr), Y_n(kr), I_n(kr), K_n(kr)$. Here $k \equiv R(\rho h\omega^2/D)^{1/4}$ where ρ is the mass density and D is the flexural rigidity [2]. The normalized radial bending moment is

$$M(r) = w''(r) + v\left[\frac{1}{r}w'(r) - \frac{n^2}{r^2}w(r)\right] \tag{1}$$

and the normalized effective shear force is

$$V(r) = w'''(r) + \frac{1}{r}w''(r) - [1 + n^2(2 - v)]\frac{1}{r^2}w'(r) + n^2(3 - v)\frac{1}{r^3}w(r). \tag{2}$$

The boundary conditions considered are clamped (C), simply supported (S), or free (F). There are therefore nine combinations of boundary conditions for the inner and the outer edges. For nontrivial solutions, an exact characteristic equation is obtained. The frequency is then determined by a simple root finder algorithm. The asymptotic form is obtained by expansions of the Bessel functions in the characteristic equation. For small $\zeta = kr$ [12],

$$J_n(\zeta) \sim \left(\frac{\zeta}{2}\right)^n \left[\frac{1}{n!} - \frac{\zeta^2}{4(n+1)!} + O(\zeta^4)\right],$$

$$I_n(\zeta) \sim \left(\frac{\zeta}{2}\right)^n \left[\frac{1}{n!} + \frac{\zeta^2}{4(n+1)!} + O(\zeta^4)\right],$$

$$Y_0(\zeta) \sim \frac{2}{\pi}\left[\gamma + \ln\left(\frac{\zeta}{2}\right)\right] + \frac{1}{2\pi}\left[1 - \gamma - \ln\left(\frac{\zeta}{2}\right)\right]\zeta^2 + O(\zeta^4),$$

$$Y_1(\zeta) \sim -\frac{2}{\pi\zeta} + \frac{1}{2\pi}\left[-1 + 2\gamma + 2\ln\left(\frac{\zeta}{2}\right)\right]z + O(\zeta^3),$$

$$Y_n(\zeta) \sim -\frac{1}{\pi}\left(\frac{2}{\zeta}\right)^n\left[(n-1)! + (n-2)!\frac{\zeta^2}{4} + O(\zeta^4)\right], \quad n \geqslant 2, \tag{3}$$

$$K_0(\zeta) \sim -\gamma - \ln\left(\frac{\zeta}{2}\right) + \frac{1}{4}\left[1 - \gamma - \ln\left(\frac{\zeta}{2}\right)\right]\zeta^2 + O(\zeta^4),$$

$$K_1(\zeta) \sim \frac{1}{\zeta} + \frac{1}{4}\left[-1 + 2\gamma + 2\ln\left(\frac{\zeta}{2}\right)\right]\zeta + O(\zeta^3),$$

$$K_n(\zeta) \sim \frac{1}{2}\left(\frac{2}{\zeta}\right)^n\left[(n-1)! - (n-2)!\frac{\zeta^2}{4} + O(\zeta^4)\right], \quad n \geqslant 2,$$

where γ is the Euler constant 0.5772. The algebra can be facilitated by a computer software with symbolic capabilities, for example Mathematica or Maple.

3. Results

First consider an annular plate where its inner edge is clamped while its outer edge is free (i.e. CF case). The boundary conditions are

$$w(b) = 0, \quad w'(b) = 0, \tag{4}$$

$$M(1) = 0, \quad V(1) = 0. \tag{5}$$

The characteristic equation is complicated but exact. For small b, the asymptotic expansions of the $n=1$ mode give

$$k \sim \frac{2}{\{|\ln b| - (23 + 9v)/[8(3 + v)]\}^{1/4}}. \tag{6}$$

The foregoing formula is more accurate than that obtained by Southwell [11]

$$k \sim \frac{2}{|\ln b|^{1/4}}. \tag{7}$$

The Poisson ratio ranges from 0.2 for concrete to 0.3 for metals to 0.4 for some polymers. Adopting the Poisson ratio $v = 0.3$, Fig. 1 shows the frequency factor k for the $n=1$ mode rising sharply from zero, and it gives the fundamental frequency for lower b values. For $b=0.2$ the fundamental frequency factor is 2.191, 2.194, 2.197 for $v = 0.2$, 0.3, 0.4, respectively. The frequency factor changes only marginally with respect to the Poisson ratio. However, the transition of the fundamental frequency to the $n=0$ mode is more affected by v due to the small slope differences as shown in Table 1.

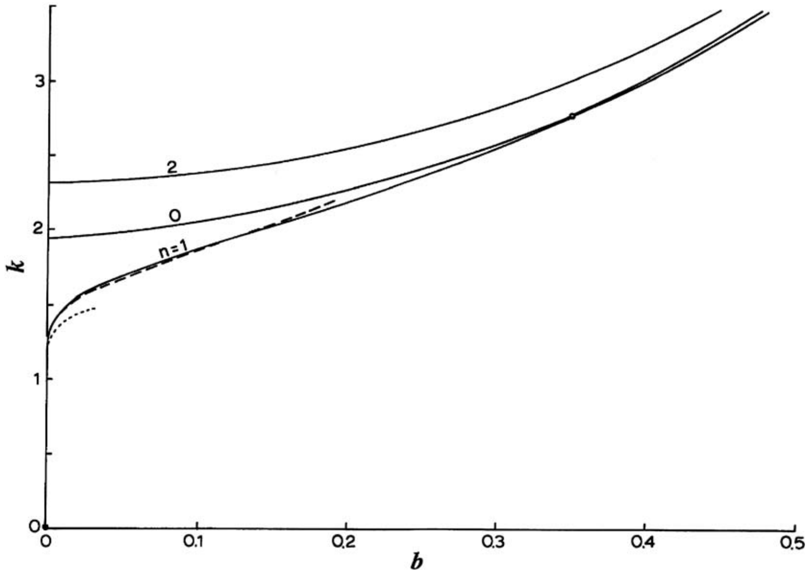

Fig. 1. Frequencies for the CF case, $v = 0.3$. Open circle represents the transition location for the fundamental modes. The dashed line is from Eq. (6) and the dotted line is from Eq. (7).

Table 1
Core radius b and fundamental frequency factor k at mode switching (from $n=1$ mode to $n=0$ mode) for CF-annular plates

	$v = 0.2$	$v = 0.3$	$v = 0.4$
b	0.307	0.349	0.409
k	2.558	2.814	3.075

Next, we consider the annular plate with its inner edge simply supported, while its outer edge is free (i.e. SF case). The boundary conditions are given by

$$w(b) = 0, \quad M(b) = 0 \tag{8}$$

and Eqs. (5). Asymptotic expansion of the characteristic equation for small values of core radius b for the $n=1$ mode gives

$$k \sim \frac{2}{\{|\ln b| + (1 + 22v + 9v^2)/[8(3 - 2v - v^2)]\}^{1/4}}. \tag{9}$$

Fig. 2 shows the fundamental frequency factor also rising singularly form zero. The transition to the $n=0$ mode is given in Table 2.

Fig. 3 shows the results for annular plates with the inner edge simply supported, while the outer edge is clamped (i.e. SC case). For small b, the fundamental frequency is also governed by the $n=1$ mode. When $b=0$, its value is 4.611, independent of v, and is obtained from the first root of

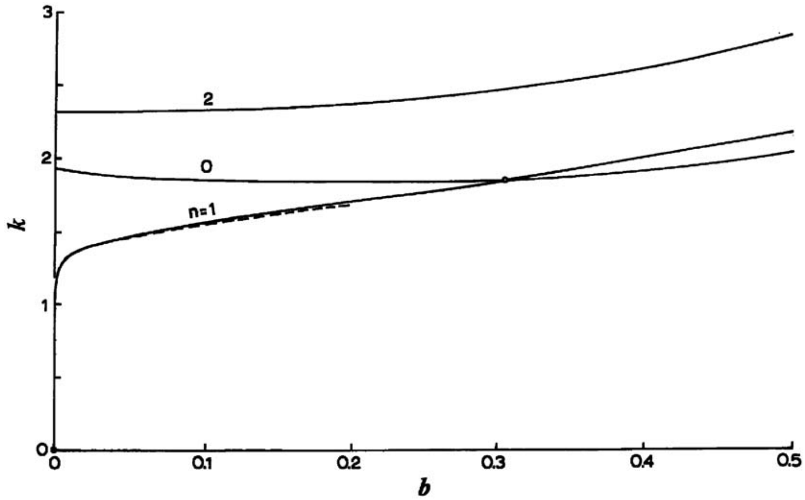

Fig. 2. Frequencies for the SF case, $v = 0.3$. The open circle represents the transition location. The dashed line is from Eq. (9).

Table 2
Core radius b and fundamental frequency factor k at mode switching (from $n=1$ mode to $n=0$ mode) for SF-annular plates

	$v = 0.2$	$v = 0.3$	$v = 0.4$
b	0.293	0.313	0.332
k	1.869	1.856	1.832

the equation

$$J_1(k)[I_0(k) + I_2(k)] - I_1(k)[J_0(k) - J_2(k)] = 0. \tag{10}$$

As b increases, the frequency k rises singularly as $|\ln b|^{-1}$ and crosses the $n=0$ mode at a location given in Table 3.

Notice the transition point also gives a local maximum for fundamental frequency.

In Fig. 4, the results for annular plates with both edges clamped (i.e. CC case) are presented. The $n=0$ mode rises from 4.611, governed by Eq. (10). The transition point is at $b=0.0132$, and $k=4.769$ independent of the Poisson ratio.

Fig. 5 shows the results for annular plates with both edges simply supported (i.e. the SS case). At $b=0$ the $n=1$ mode rises as $|\ln b|^{-1}$ from the first root of the equation

$$\{2v[I_0(k) + I_2(k)] + kI_3(k)\}J_1(k) - \{2v[J_0(k) - J_2(k)] - k[6J_1(k) - J_3(k)]\}I_1(k) = 0. \tag{11}$$

The fundamental frequencies are 3.711, 3.728 and 3.745 for $v = 0.2$, 0.3, 0.4, respectively. The transition core radii and the corresponding frequency factors are given in Table 4.

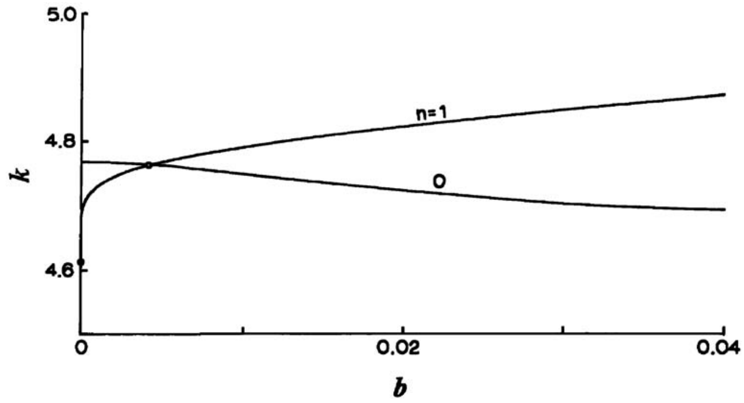

Fig. 3. Frequencies for the SC case, $v = 0.3$. The open circle represents the transition location. The solid circle is the fundamental frequency at $b = 0$.

Table 3
Core radius b and fundamental frequency factor k at mode switching (from $n = 1$ mode to $n = 0$ mode) for SC-annular plates

	$v = 0.2$	$v = 0.3$	$v = 0.4$
b	0.0036	0.0042	0.0048
k	4.763	4.762	4.758

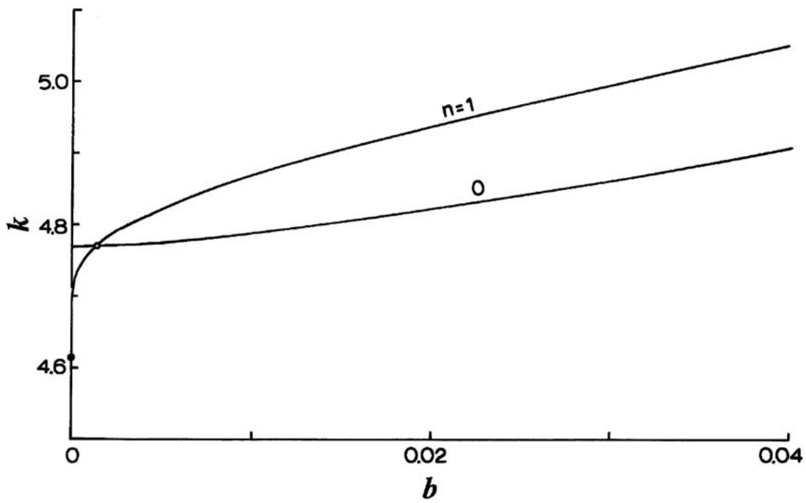

Fig. 4. Frequencies for the CC case. The open circle represents the transition location. The solid circle is the fundamental frequency at $b = 0$.

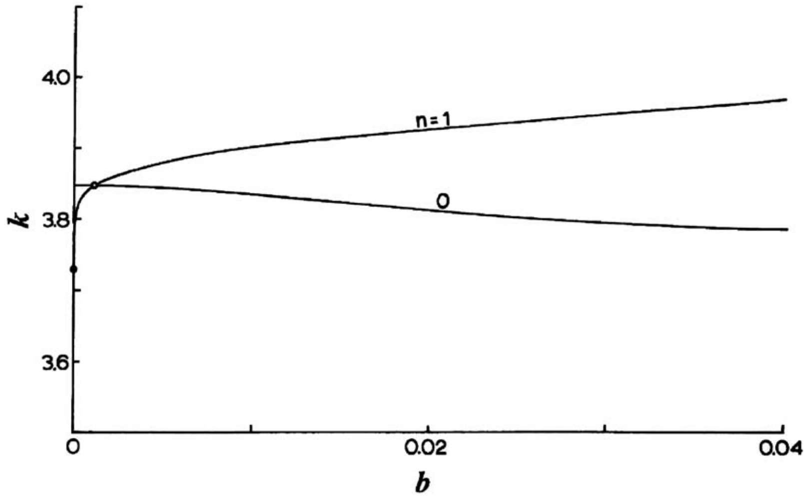

Fig. 5. Frequencies for the SS case, $v = 0.3$. The open circle represents the transition location. The solid circle is the fundamental frequency at $b = 0$.

Table 4
Core radius b and fundamental frequency factor k at mode switching (from $n = 1$ mode to $n = 0$ mode) for SS-annular plates

	$v = 0.2$	$v = 0.3$	$v = 0.4$
b	0.0011	0.0013	0.0018
k	3.831	3.848	3.864

Fig. 6 shows the results for annular plates with inner edge clamped, while the outer edge simply supported (i.e. CS case). The start of the $n = 1$ mode is same as the SS annular plate case, but the transition is much closer to $b = 0$ as shown in Table 5.

When the inner edge is free, the small core has little effect on the frequency, which is the frequency of the full circular plate. Thus at $b = 0$, the fundamental frequency factor for the FC case is 3.196 for all v values, which is obtained from the $n = 0$ mode and

$$I_1(k)J_0(k) + I_0(k)J_1(k) = 0. \tag{12}$$

For the FS case the characteristic equation for the axisymmetric mode (i.e. $n = 0$) is

$$[2vI_1(k) + kI_2(k)]J_0(k) + [2kJ_0(k) + 2vJ_1(k) - kJ_2(k)]I_0(k) = 0. \tag{13}$$

The values of the fundamental frequency factor k at $b = 0$ is 2.187, 2.222, 2.253 for $v = 0.2$, 0.3, 0.4, respectively.

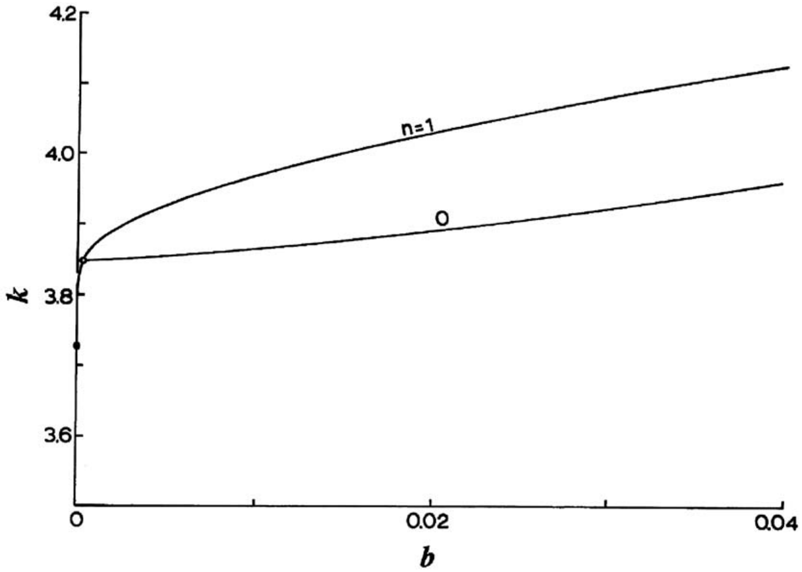

Fig. 6. Frequencies for the CS case, $\nu = 0.3$. The open circle represents the transition location. The solid circle is the fundamental frequency at $b = 0$.

Table 5
Core radius b and fundamental frequency factor k at mode switching (from $n = 1$ mode to $n = 0$ mode) for CS-annular plates

	$\nu = 0.2$	$\nu = 0.3$	$\nu = 0.4$
b	0.00032	0.00034	0.00036
k	3.831	3.849	3.866

For the FF case the $n = 2$ mode gives the fundamental frequency of 2.378, 2.315, 2.241 for $\nu = 0.2$, 0.3, 0.4, respectively. Note that F–F annular plates are used in modeling very large, pontoon-type floating structures and flying disks.

4. Discussions

The fundamental frequencies for annular plates with small core sizes are now determined. Asymptotic expansion on the exact characteristic equations delineates the singular rise (infinite slope) of the fundamental frequency when the core size is close to zero. These properties cannot be obtained by *any* numerical solution of the vibration equations due to severe scaling problems for minute cores.

Except when the inside boundary is free, the fundamental frequencies are always governed by the $n = 1$ mode for small b, and the $n = 0$ mode for larger b. Since the frequency for the $n = 1$ mode rises singularly, the decrease in fundamental frequency may be quite significant for small cores. Our figures and tables would be useful in the design of vibrating annular plates with small cores.

For $b=0$, the fundamental frequency is zero for CF and SF cases. This represents a rigid rotation about a diameter. Our approximate solutions given by Eqs. (6) and (9) are quite accurate for $b<0.1$. For the SC, CC, SS, CS cases the fundamental frequencies are finite when $b=0$ but the rise is still singular. Singular rise is absent for the FC, FS, FF cases.

The transition from $n=1$ to 0 modes may occur at very small b values in some cases. In order that the plate equations remain valid, the thickness should be much less than the core radius. For example, consider a plate of outer diameter 1 m. If $b=0.01$ the core has a diameter of 1 cm. For the thin plate equation to be valid the plate thickness should be less than 1 mm.

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